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On head-on collisions between two solitary waves

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We consider a head-on collision between two solitary waves on the surface of an inviscid homogeneous fluid. A perturbation method which in principle can generate an asymptotic series of all orders, is used to calculate the effects of the collision. We find that the waves emerging from (i.e. long after) the collision preserve their original identities to the third order of accuracy we have calculated. However a collision does leave imprints on the colliding waves with phase shifts and shedding of secondary waves. Each secondary wave group trails behind its primary, a solitary wave. The amplitude of the wave group diminishes in time because of dispersion. We have also calculated the maximum run-up amplitude of two colliding waves. The result checks with existing experiments.

1. Introduction

We recast, in § 2, the equations of motion of an inviscid, constant-density fluid with a free surface into a pair of equations in terms of the free surface elevation $h(t,x,y)$ and the velocity along a horizontal stream bed $W(t,x,y)$. These equations are convenient for a perturbation scheme to be introduced in § 3 to study a head-on collision between two solitary waves which are small in amplitude ($a/h_0 \ll 1$), and long in wavelength ($\lambda/h_0 \gg 1$). Here a is the measure of amplitude, λ the wavelength, and h_0 the undisturbed depth of the fluid. The amplitude and the wavelength parameters are related by Ursell's ordering for theory of shallow water, i.e. $a\lambda^2 \approx h_0^3$. We have carried out the calculations to the third order of approximation.

In the first-order approximation, we have two independently moving solitary waves $aS(\xi)$ and $bS(\eta)$. $S(x) \equiv \operatorname{sech}^2 \frac{1}{2}x$, a progressive wave of permanent type which satisfies the Korteweg–de Vries equation. The variables ξ and η denote the right- and left-going wave-framed co-ordinates respectively. The constants a and b specify the heights of the waves. In the second-order approximation, we find that the wave field is modified by (1) quadratic terms in S, (2) change of wave speeds and (3) addition of phase functions $\theta_0(\eta)$ and $\phi_0(\xi)$ to ξ and η respectively. For two waves at large separation, i.e. before or after collision, the first two corrections above reduce to Laitone's (1960) second-order calculation of a single solitary wave. On the other hand, we can also use the result in (1) to calculate the maximum run-up amplitude during the collision. Our result checks with an earlier calculation of Byatt-Smith (1971). Since, up to this order of accuracy, the phase function θ (or ϕ) is a function of η (or ξ) alone, and differs by a constant value through a collision, this value represents a phase shift for a right- (or left-) going solitary wave. This result agrees with that of Oikawa & Yajima (1973). In the third-order approximation, we obtain, as before, those three

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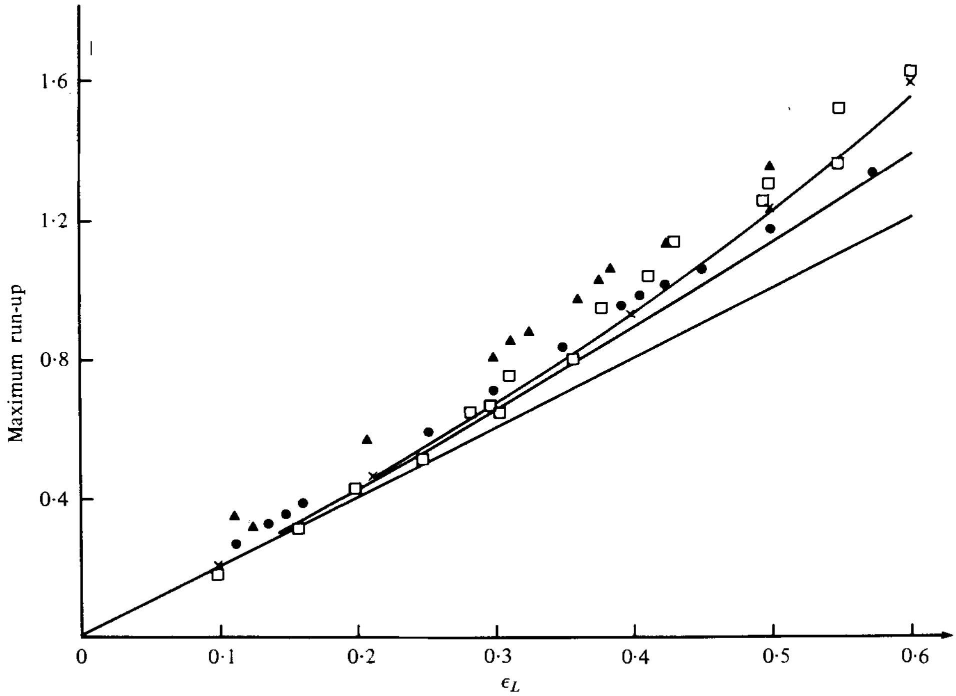

FIGURE 1. Maximum run-up ζ ($f = 1, g = 1$) *vs.* wave amplitude $\Delta h/h_0 = \epsilon_L$: 1, first order; 2, second order; 3, third order. △, Maxworthy's (1976) wave–wave interaction; ○, end-wall reflexion; ×, Chan & Street (1970) numerical results; □, experimental results.

corrections to the wave field. The first correction is a polynomial of third degree in S. This polynomial in S and the wave speed correction reduce as the separation between waves increases, to the known result of Chappelear (1962), Grimshaw (1971) and Fenton (1972) pertaining to a single solitary wave with third-order accuracy. However, the phase functions obtained become functions both of ξ and η. This gives different values of phase shifts at different points in a wave. Unlike the uniform phase shift in the second-order approximation which preserves the wave form, here these variable phase shifts cause a distortion of the wave form in addition to a simple translation. Since this distorted wave field does not satisfy the equation for waves which are propagated without a change in shape and speed, we are forced to investigate, in § 5, the slow time evolution of this distorted wave forms. We find that the waves eventually transform back to their original forms with shedding of secondary wavelets. These secondary waves propagate with diminishing amplitude in the direction opposite to that of the main waves. Thus we can still speak of preservation of wave form up to the third order of accuracy, with the understanding that there exists uniform phase shifts as well as shedding of secondary waves. We stop our calculation at the third order. However we feel, after working long with the perturbation scheme used here, that the higher-order results will be essentially of similar nature. There are two imprints of a collision between two solitary waves: uniform phase shifts and shedding of secondary waves. We have also calculated the maximum run-up amplitude up to the third order of accuracy for two colliding solitary waves. The result agrees very closely with the numerical and experimental results of Chan & Street (1970). It also checks with Maxworthy's (1976) experiment for a solitary wave reflected from a wall (see figure 1).

Head-on collisions between two solitary waves using different perturbation methods have been studied by various authors obtaining results up to second-order accuracy. Miles (1977) calls this kind of collision weak interaction in contrast to those intrinsically nonlinear overtaking interactions. We note that the latter problem was brilliantly solved by Gardner *et al.* (1967) using the Korteweg–de Vries equation. For a brief historical account on weak interactions, we refer the reader to the work of Miles (1977).

2. Basic Equations

We take the velocity field of our flow problem to be described by a potential $\phi(t,x,y,z)$, satisfying the Laplace equation. The free surface, where the pressure vanishes, is specified by an unknown function $z = h(t,x,y)$. The fluid is supported by a horizontal plane at $z = 0$, where we set the normal velocity $\partial\phi/\partial z = 0$. It is easy to show in this case that the potential ϕ can be expressed as a Taylor series at $z = 0$. Using $\nabla^2\phi = 0$ for $z > 0$ and $\partial\phi/\partial z = 0$ at $z = 0$, we obtain

$$\phi(t,x,y,z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \nabla^{2n}\Phi, \tag{1}$$

where

$$\Phi(t,x,y) = \phi(t,x,y,z=0), \quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y},$$

representing a gradient operator in the x, y plane. We can now express the kinematic condition at the free surface as

$$\partial h/\partial t + \nabla\phi . \nabla h = \partial\phi/\partial z, \tag{2}$$

and the Bernoulli theorem also applied at the free surface as

$$\partial\phi/\partial t + gh + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = C(t) \tag{3}$$

(see Whitham 1974, ch. 13).

In terms of Φ these become

$$\frac{\partial h}{\partial t} + \nabla . \left[\sum_{n=0}^{\infty} (-1)^n \frac{h^{2n+1}}{(2n+1)!} \nabla^{2n}(\nabla\Phi)\right] = 0, \tag{4}$$

$$\frac{\partial\Phi}{\partial t} + gh + \tfrac{1}{2}(\nabla\Phi)^2 + \sum_{n=1}^{\infty} (-1)^n \frac{h^{2n}}{(2n)!} \left[\nabla^{2n}\Phi_t + \tfrac{1}{2}\sum_{m=0}^{2n} (-1)^m \binom{2n}{m} \nabla^{m+1}\Phi * \nabla^{2n-m+1}\Phi\right] = C(t), \tag{5}$$

where the asterisk stands for the vector inner product for even m and the usual arithmetic multiplication for odd m, and

$$\binom{2n}{m} = \frac{(2n)!}{m!\,(2n-m)!},$$

the binomial coefficients. We shall consider waves in a channel. We therefore drop the

y dependence in h and Φ. By taking the x derivative of (5), we obtain in place of (4) and (5) the following two equations:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left\{hw+\sum_{n=1}^{\infty}(-1)^n\frac{h^{2n+1}}{(2n+1)!}\frac{\partial^{2n}w}{\partial x^{2n}}\right\}=0, \tag{6}$$

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial x}\left\{gh+\frac{w^2}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{h^{2n}}{(2n)!}\left[\frac{\partial^{2n}w}{\partial t\,\partial x^{2n-1}}\right.\right.$$
$$\left.\left.+\tfrac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\frac{\partial^m w}{\partial x^m}\frac{\partial^{2n-m}w}{\partial x^{2n-m}}\right]\right\}=0, \tag{7}$$

where $w = \partial\Phi/\partial x$ represents the velocity at the bottom of the channel.

Both of the above equations are in the conservative form. It is interesting to point out that two more conservation laws can be derived from (6) and (7). They represent conservation of horizontal momentum and total energy, i.e.

$$\frac{\partial}{\partial t}\left(\int_0^h u\,dz\right)+\frac{\partial}{\partial x}\left[\int_0^h\left(u^2+\frac{p}{\rho}\right)dz\right]=0, \tag{8}$$

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}gh^2+\frac{1}{2}\int_0^h(u^2+v^2)\,dz\right]+\frac{\partial}{\partial x}\int_0^h dz\,u\left[\tfrac{1}{2}(u^2+v^2)+\frac{p}{\rho}+gz\right]=0, \tag{9}$$

where

$$\frac{p}{\rho}=g(h-z)+\int_z^h\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial z}\right)v\,dz,$$

and $u=\phi_x$ and $v=\phi_z$ are both expressed in w by (1). We shall however in this paper use (6) and (7) only.

3. Perturbation solution for two solitary waves colliding head-on

Consider two solitary waves, far apart, of small but finite amplitude and heading towards each other. The time evolution of their interaction, and the final state after their collision will be our main concern.

We introduce the following co-ordinates transformations (wave frames)

$$\xi_0=\epsilon^{\frac{1}{2}}k(x-C_Rt), \tag{10}$$
$$\eta_0=\epsilon^{\frac{1}{2}}l(x+C_Lt),$$

where $0<\epsilon\ll 1$, ϵ is a dimensionless parameter representing the order of magnitude of the wave amplitude. The scaling of the horizontal wavelength in accordance with Ursell's relationship is taken as $\epsilon^{\frac{1}{2}}$, leaving k and l as the wavenumbers of order unity for the right- and left-going waves respectively. The right- and left-going wave speeds C_R, C_L are to be related to the amplitudes of the waves. In the limit of infinitesimal amplitude, they take the value of linear wave speed $(gh_0)^{\frac{1}{2}}$. Anticipating that a difficulty might show up in our perturbation method, we introduce the following transformations of wave-framed co-ordinates with phase functions†

† This is essentially the method of strained co-ordinates first introduced by Poincaré (1892) for ordinary differential equations and later generalized to hyperbolic partial differential equations by Lighthill (1949) and Lin (1954). For a detailed discussion we refer the reader to Van Dyke (1964).

$$\xi_0 = \xi - \epsilon k\theta(\xi,\eta), \qquad (11)$$
$$\eta_0 = \eta - \epsilon l\phi(\xi,\eta),$$

where $\theta(\xi,\eta)$ and $\phi(\xi,\eta)$ are to be determined in the process of our perturbational solution of (6) and (7). These functions, introduced for the purpose of making asymptotic approximations, allow us to calculate phase changes due to collision.

Using (10) and (11) we obtain the transformation between derivatives as

$$\frac{\partial}{\partial t}+C_R\frac{\partial}{\partial x}=\frac{\epsilon^{\frac{3}{2}}}{D}(C_R+C_L)\left[l\frac{\partial}{\partial \eta}+\epsilon kl\left(\frac{\partial\theta}{\partial\eta}\frac{\partial}{\partial\xi}-\frac{\partial\theta}{\partial\xi}\frac{\partial}{\partial\eta}\right)\right], \qquad (12)$$

$$\frac{\partial}{\partial t}-C_L\frac{\partial}{\partial x}=-\frac{\epsilon^{\frac{3}{2}}}{D}(C_R+C_L)\left[k\frac{\partial}{\partial \xi}+\epsilon kl\left(\frac{\partial\phi}{\partial\xi}\frac{\partial}{\partial\eta}-\frac{\partial\phi}{\partial\eta}\frac{\partial}{\partial\xi}\right)\right], \qquad (13)$$

where

$$D=\left(1-\epsilon k\frac{\partial\theta}{\partial\xi}\right)\left(1-\epsilon l\frac{\partial\phi}{\partial\eta}\right)-\epsilon^2 kl\frac{\partial\theta}{\partial\eta}\frac{\partial\phi}{\partial\xi}.$$

Setting $h=h_0(1+\zeta)$, we rewrite (6) and (7) into

$$\left[\frac{\partial}{\partial t}\pm C_{R,L}\frac{\partial}{\partial x}\right][w\pm C\zeta]+\frac{\partial}{\partial x}F_{\pm}=0, \qquad (14)$$

where $C=(gh_0)^{\frac{1}{2}}$ is the linear wave speed, and

$$\begin{aligned}F_{\pm}=&\pm(C-C_{R,L})(w\pm C\zeta)+\frac{w^2}{2}\pm C\zeta w\\&+\sum_{n=1}^{\infty}(-1)^n\frac{h_0^{2n}(1+\zeta)^{2n}}{(2n)!}\left[\frac{\partial^{2n}w}{\partial t\,\partial x^{2n-1}}\pm\frac{C(1+\zeta)}{2n+1}\frac{\partial^{2n}w}{\partial x^{2n}}\right.\\&\left.+\tfrac{1}{2}\sum_{n=0}^{2n}(-1)^m\binom{2n}{m}\frac{\partial^m w}{\partial x^m}\frac{\partial^{2n-m}w}{\partial x^{2n-m}}\right].\end{aligned} \qquad (15)$$

It is convenient to make the following change of dependent variables

$$w+C\zeta=2\epsilon C\alpha,\quad w-C\zeta=-2\epsilon C\beta, \qquad (16)$$

which give us $w=\epsilon c(\alpha-\beta)$ and $\zeta=\epsilon(\alpha+\beta)$. Together with (12) and (13) we can now represent (14) as

$$\begin{aligned}2C\epsilon(C_R+C_L)&\left[l\frac{\partial\alpha}{\partial\eta}+\epsilon kl\left(\theta_\eta\frac{\partial\alpha}{\partial\xi}-\theta_\xi\frac{\partial\alpha}{\partial\eta}\right)\right]\\&+\left\{k\frac{\partial}{\partial\xi}+l\frac{\partial}{\partial\eta}+\epsilon kl\left[\frac{\partial}{\partial\eta}(\theta-\phi)\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\xi}(\theta-\phi)\frac{\partial}{\partial\eta}\right]\right\}F_+=0.\end{aligned} \qquad (17)$$

A similar equation for β is obtained by replacing α by β, ξ by η, k by l, F_+ by F_- and θ by ϕ. We need hereafter consider (17) only. We now express the new variables in the following power series

$$\left.\begin{aligned}\alpha(\xi,\eta)&=\alpha_0+\epsilon\alpha_1+\epsilon^2\alpha_2+\dots,\\\beta(\xi,\eta)&=\beta_0+\epsilon\beta_1+\epsilon^2\beta_2+\dots,\\\theta(\xi,\eta)&=\theta_0(\eta)+\epsilon\theta_1(\xi,\eta)+\dots,\\\phi(\xi,\eta)&=\phi_0(\xi)+\epsilon\phi_1(\xi,\eta)+\dots,\\C_R&=C(1+\epsilon aR_1+\epsilon^2a^2R_2+\epsilon^3a^3R_3+\dots),\\C_L&=C(1+\epsilon bL_1+\epsilon^2b^2L_2+\epsilon^3b^3L_3+\dots).\end{aligned}\right\} \qquad (18)$$

Substituting (18) into (17), we obtain a lengthy expression in power series of ϵ (see appendix). The coefficients of ϵ, ϵ^2, ϵ^3 and part of ϵ^4 will, however, be presented in sequel as follows.

(i) *Coefficients of* ϵ

The equations are simply

$$\frac{\partial \alpha_0}{\partial \eta} = 0, \quad \frac{\partial \beta_0}{\partial \xi} = 0.$$

Their solutions are

$$\alpha_0 = af(\xi), \quad \beta_0 = bg(\eta), \tag{19}$$

where the constants a, b appearing (19) as well as in (18) are introduced to allow us to take $f(0) = g(0) = 1$; f and g are arbitrary functions to be determined next.

(ii) *Coefficients of* ϵ^2

The equation for α_1 takes the form

$$4l\left(\frac{\partial \alpha_1}{\partial \eta} + ak\frac{\partial \theta_0}{\partial \eta}f'\right) - 2R_1a^2kf' + 3a^2kff'' - b^2lgg' - ablfg' - abkf'g + \frac{h_0^2}{3}(ak^3f''' + 2bl^3g''') = 0. \tag{20}$$

The terms appearing in (20) can be grouped into three kinds:

(*a*) *Secular terms:* Those which are independent of η. There are three of them in (20): $ka(-2R_1af' + 3aff' + \frac{1}{3}h_0^2k^2f''')$. Upon integrating these with respect to η, we obtain secular behaviour, i.e. a contribution which becomes unbounded in time or space. We set these secular terms equal to zero and obtain, after letting

$$R_1 = \tfrac{1}{2}, \quad h_0^2k^2 = 3a, \tag{21}$$

an equation which f has to satisfy, viz.

$$f''' + 3ff' - f' = 0, \quad \text{or} \quad f = S(\xi) = \operatorname{sech}^2(\tfrac{1}{2}\xi). \tag{22}$$

Similarly we have, from the equation for β,

$$L_1 = \tfrac{1}{2}, \quad h_0^2l^2 = 3b, \quad g = S(\eta) = \operatorname{sech}^2(\tfrac{1}{2}\eta). \tag{23}$$

(*b*) *Non-local terms.* These are not secular in the present, they will be if left as they are. The solution of α_1 due to them comes under an integral sign. Physically this represents the memory of a collision. In (20), we identify the following two terms to be in this category

$$4kal\frac{\partial \theta_0}{\partial \eta}f' - abkgf'.$$

We again set this equal to zero, and solve for θ_0 as

$$\theta_0 = \frac{b}{4l}\int_{-\infty}^{\eta} g(\eta_1)\,d\eta_1. \tag{24}$$

Similarly we have

$$\phi_0 = \frac{a}{4k}\int_{+\infty}^{\xi} f(\xi_1)\,d\xi. \tag{25}$$

(The above result agrees with an earlier work by Oikawa & Yajima 1973.)

	Before collision	After collision
Right-going wave f	$\eta \to -\infty$	$\eta \to +\infty$
Left-going wave g	$\xi \to +\infty$	$\xi \to -\infty$

TABLE 1

The choice of the lower limits of integrations will be explained in the next paragraph. Note that the term $abkgf'$ as mentioned earlier is non-secular at this order. Had we let $\theta_0 = 0$ we could have solved α_1 with a term of the form $f' \int^{\eta} g(\eta_1)\, d\eta_1$. Such an α_1 will cause a secular solution in α_2.

We now identify the right- and left-going waves as those described by f and g respectively. We shall specify the location of these waves before and after collision in terms of asymptotic values of ξ and η in table 1. We can then calculate from (24) and (25) the shifts of the waves after collision. These phase shifts make a uniform translation of the ξ and η while leaving the profiles intact. We shall call these simple phase shifts.

(*c*) *Local terms*. The remaining terms in (20) can be integrated to give

$$\alpha_1(\xi, \eta) = \tfrac{7}{8}b^2g^2 + \frac{ab}{4}fg - \frac{b^2}{2}g + a^2F_1(\xi), \tag{26}$$

similarly

$$\beta_1(\xi, \eta) = \tfrac{7}{8}a^2f^2 + \frac{ab}{4}fg - \frac{a^2}{2}f + b^2G_1(\eta), \tag{27}$$

where $F_1(\xi)$ and $G_1(\eta)$ are two arbitrary functions to be determined from the consideration of the secular terms in the next order of approximation.

(iii) *Coefficients of* ϵ^3

The number of terms in this case is already rather formidable: we shall list them separately in the different groups only.

(*a*) *Secular terms:* Setting these equal to zero, we have

$$F_1'' + (3f-1)F_1 = (2R_2 - \tfrac{19}{20})f + \tfrac{9}{8}f^2 + \tfrac{1}{4}f^3. \tag{28}$$

The first term on the right-hand side of (28) gives a solution of F_1 as

$$(2R_2 - \tfrac{19}{20})(f + \tfrac{1}{2}\xi f'),$$

which is unbounded as $\xi \to \pm\infty$, i.e. the series solution is not asymptotic. We thus require the coefficient of this term to vanish, i.e.

$$R_2 = \tfrac{19}{40}. \tag{29}$$

This gives a second-order correction for the wave speed.

The remaining equation can be readily solved and we have

$$F_1 = -\tfrac{1}{8}f^2 + f. \tag{30}$$

We have now the complete solution for α_1 and β_1:

$$\alpha_1(\xi,\eta) = \tfrac{1}{8}(7b^2g^2 - a^2f^2) - \tfrac{1}{2}(b^2g - 2a^2f) + \frac{ab}{4}fg, \tag{31}$$

$$\beta_1(\xi,\eta) = \tfrac{1}{8}(7a^2f^2 - b^2g^2) - \tfrac{1}{2}(a^2f - 2b^2g) + \frac{ab}{4}fg. \tag{32}$$

To the result of (30) we can also add one of the bounded homogeneous solutions of (28), which is the first derivative of f. However, as we go to higher order, it is seen that such an addition only amounts to a uniform shift of the origin of ξ which represents a simple phase shift defined above. We will henceforth drop it entirely.

(*b*) *Non-local terms.* These terms will give the solution of θ_1 and ϕ_1 as

$$\theta_1(\xi,\eta) = \frac{9ab}{4l}f\int_{-\infty}^{\eta} g(\eta_1)\,d\eta_1 + \frac{1}{4l}\int_{-\infty}^{\eta}\left[\frac{3b^2}{2}g^2 - \left(\frac{13a}{4}+\frac{b}{4}\right)bg\right]d\eta_1, \tag{33}$$

$$\phi_1(\xi,\eta) = \frac{9ab}{4k}g\int_{+\infty}^{\xi} f(\xi_1)\,d_1 + \frac{1}{4k}\int_{+\infty}^{\xi}\left(\tfrac{3}{2}a^2f^2 - \left(\tfrac{13}{4}b+\frac{a}{4}\right)af\right]d\xi_1. \tag{34}$$

Each of the second terms above is of similar nature to the first-order phase shift formulae (24) and (25). They give simple phase shifts. Note however that the first term in (33) depends on ξ as $\eta \to +\infty$. Since θ_1 enters into the argument ξ of the function f, we see that the wave profile of the right-going solitary wave differs from the one before collision. It tilts backward to the direction of propagation of the wave. (See figure 2 and the discussion in (ii) below.) Similar behaviour appears in the left-going wave because of the first term in (34).

In § 5 we will study the time evolution of these unsymmetrical waves and show that its unsymmetrical part, acquired during collision, will separate and propagate away from the symmetric waves of permanent type which are identical to the waves before collision.

(*c*) *Local terms.* After integrating the local terms, we obtain

$$\alpha_2 = \tfrac{1}{32}b^3g^3 + 2ab^2fg^2 - \tfrac{1}{4}a^2bf^2g - \tfrac{9}{8}ab^2fg + \tfrac{3}{8}a^2bfg + \tfrac{43}{32}b^3g^2 - \tfrac{7}{10}b^3g + a^3F_2(\xi). \tag{35}$$

A similar expression is obtained for β_2, viz.

$$\beta_2 = \tfrac{1}{32}a^3f^3 + 2a^2bf^2g - \tfrac{1}{4}ab^2fg^2 - \tfrac{9}{8}a^2bfg + \tfrac{3}{8}ab^2fg + \tfrac{43}{32}a^3f^2 - \tfrac{7}{10}a^3f + b^3G_2(\eta). \tag{36}$$

The arbitrary functions to integration $F_2(\xi)$ and $G_2(\eta)$ like its counterpart in the previous order of approximation must be determined from the secularity-free condition in the next order of approximation.

(iv) *Coefficients of* ϵ^4

We shall focus only on the secular terms in this order. Setting them equal to zero, we obtain an equation for F_2 similar to (28) as follows

$$F_2'' + (3f-1)F_2 = (2R_3 - \tfrac{55}{65})f - \tfrac{393}{160}f^2 + \tfrac{201}{16}f^3 - \tfrac{591}{64}f^4. \tag{37}$$

The above equation is solved by

$$R_3 = \tfrac{55}{112} \quad \text{and} \quad F_2(\xi) = \tfrac{197}{160}f^3 - \tfrac{217}{160}f^2 + \tfrac{43}{40}f. \tag{38}$$

Similarly

$$L_3 = \tfrac{55}{112} \quad \text{and} \quad G_2(\eta) = \tfrac{197}{160}g^3 - \tfrac{217}{160}g^2 + \tfrac{43}{40}g \tag{39}$$

for the left-going wave. This completes our result for α_2 and β_2 as follows

$$\alpha_2 = \tfrac{1}{32}b^3g^3 + 2ab^2fg^2 - \tfrac{1}{4}a^2bf^2g - \tfrac{9}{8}ab^2fg + \tfrac{3}{8}a^2bfg + \tfrac{43}{32}b^3g^2 - \tfrac{7}{10}b^3g + \tfrac{43}{40}a^3f - \tfrac{217}{160}a^3f^2 + \tfrac{197}{160}a^3f^3, \tag{40}$$

and

$$\beta_2 = \tfrac{1}{32}a^3f^3 + 2a^2bfg^2 - \tfrac{1}{4}ab^2fg^2 - \tfrac{9}{8}a^2bfg + \tfrac{3}{8}ab^2fg + \tfrac{43}{32}a^3f^2 - \tfrac{7}{10}a^3f + \tfrac{43}{40}b^3g + \tfrac{217}{160}b^3g^3 + \tfrac{197}{160}b^3g^3. \tag{41}$$

4. Summary of results of section 3

We have obtained, in § 3, the following results for the velocity at the bottom, w, the perturbed free surface elevation ζ, the wave speeds C_R, C_L, and the phase functions θ, ϕ:

$$w(f,g) = \epsilon C\{af - bg + \epsilon[-a^2f^2 + b^2g^2 + \tfrac{3}{2}(a^2f - b^2g)] + \epsilon^2[\tfrac{6}{5}(a^3f^3 - b^3g^3) - \tfrac{27}{10}(a^3f^2 - b^3g^2) + \tfrac{71}{40}(a^3f - b^3g) + \tfrac{3}{2}(a^2bfg - ab^2fg) - \tfrac{9}{4}(a^2bf^2g - ab^2fg^2)]\} \tag{42}$$

and

$$\zeta(f,g) = \epsilon\{af + bg + [\tfrac{3}{4}(a^2f^2 + b^2g^2) + \tfrac{1}{2}abfg + \tfrac{1}{2}(a^2f + b^2g)] + \epsilon^2[\tfrac{101}{80}(a^3f^3 + b^3g^3) - \tfrac{1}{80}(a^3f^2 - b^3g^2) + \tfrac{7}{4}(ab^2fg^2 + a^2bf^2g) - \tfrac{3}{4}(a^2bfg + ab^2fg) + \tfrac{3}{8}(a^3f + b^3g)]\}, \tag{43}$$

$$\frac{C_R}{C} = 1 + \frac{\epsilon a}{2} + \tfrac{19}{40}\epsilon^2a^2 + \tfrac{55}{112}\epsilon^3a^3, \tag{44}$$

$$\frac{C_L}{C} = 1 + \frac{\epsilon b}{2} + \tfrac{19}{40}\epsilon^2b^2 + \tfrac{55}{112}\epsilon^3b^3, \tag{45}$$

$$\theta = \frac{b}{4l}\left\{\int_{-\infty}^{\eta}\left(1 + \tfrac{3}{2}\epsilon bg - \tfrac{13}{4}a\epsilon - \frac{b}{4}\epsilon\right)g\,d\eta + 9\epsilon af(\xi)\int_{-\infty}^{\eta} g\,d\eta\right\}, \tag{46}$$

$$\phi = \frac{a}{4k}\left\{\int_{+\infty}^{\xi}\left(1 + \tfrac{3}{2}\epsilon af - \tfrac{13}{4}b\epsilon - \frac{a\epsilon}{4}\right)f\,d\xi + 9\epsilon bg(\eta)\int_{+\infty}^{\xi} f\,d\xi\right\}, \tag{47}$$

where $f = \operatorname{sech}^2\frac{1}{2}\xi$, and $g = \operatorname{sech}^2\frac{1}{2}\eta$ with ξ and η as defined by (10) and (11).

To compare these results with the earlier works on one solitary wave of Laitone, Chappelear, Grimshaw and Fenton, we note that ϵ as used here is equivalent to $(L_1 - L_3)$ of Chappelear but different from a non-dimensional wave-amplitude parameter used by Laitone, Grimshaw and Fenton; the latter is defined as the ratio of maximum wave height, $a_{\max}$, to the undisturbed depth (i.e. $a_{\max}/h_0$). They expressed the wavenumber and wave speed in powers of this parameter. On the other hand we, in this paper, have fixed the wavenumber parameter as $\epsilon^{\frac{1}{2}}$ and expanded the other quantities in terms of ϵ. From (43), we see that

$$\left.\begin{aligned}\zeta(1,0) &= \epsilon a + a^2\epsilon^2(\tfrac{5}{4}) + a^3\epsilon^3(\tfrac{13}{8}) + O(\epsilon^4),\\ \text{and}\quad \zeta(0,1) &= \epsilon b + b^2\epsilon^2(\tfrac{5}{4}) + b^3\epsilon^3(\tfrac{13}{8}) + O(\epsilon^4).\end{aligned}\right\} \tag{48}$$

Defining $\epsilon_R = \zeta(1,0)$ and $\epsilon_L = \zeta(0,1)$, each of which is the amplitude parameters used by Laitone, Grimshaw and Fenton, we have after inverting the above relationship

$$\left.\begin{aligned}\epsilon a &= \epsilon_R - \tfrac{5}{4}\epsilon_R^2 + \tfrac{3}{2}\epsilon_R^3 + O(\epsilon_R^4),\\ \epsilon b &= \epsilon_L - \tfrac{5}{4}\epsilon_L^2 + \tfrac{3}{2}\epsilon_L^3 + O(\epsilon_L^4).\end{aligned}\right\} \tag{49}$$

We now rewrite (43)–(47) in terms of ϵ_R and ϵ_L as follows:

$$\begin{aligned}\zeta(\xi,\eta) = {}& \epsilon_R\{f + \tfrac{3}{4}\epsilon_R(f^2 - f) + \epsilon_R^2(\tfrac{101}{80}f^3 - \tfrac{151}{80}f^2 + \tfrac{5}{8}f)\}\\ &+ \epsilon_L\{g + \tfrac{3}{4}\epsilon_L(g^2 - g) + \epsilon_L^2(\tfrac{101}{80}g^3 - \tfrac{151}{80}g^2 + \tfrac{5}{8}g)\}\\ &+ \epsilon_R\epsilon_L\{\tfrac{1}{2} + [\tfrac{7}{4}(\epsilon_R f + \epsilon_L g) - \tfrac{11}{8}(\epsilon_R + \epsilon_L)]\}fg,\end{aligned} \tag{50}$$

$$\frac{C_R}{C} = 1 + \frac{\epsilon_R}{2} - \tfrac{3}{20}\epsilon_R^2 + \tfrac{3}{56}\epsilon_R^3, \tag{51}$$

$$\frac{C_L}{C} = 1 + \frac{\epsilon_L}{2} - \tfrac{3}{20}\epsilon_L^2 + \tfrac{3}{56}\epsilon_L^3, \tag{52}$$

$$\begin{aligned}\frac{\xi}{2} &= \frac{1}{h_0}\left(\frac{3\epsilon_R}{4}\right)^{\frac{1}{2}}(1 - \tfrac{5}{8}\epsilon_R + \tfrac{71}{128}\epsilon_R^2)\left\{x - C_R t + \frac{h_0}{4}\left(\frac{\epsilon_L}{3}\right)^{\frac{1}{2}}\left[\int_{-\infty}^{\eta}[1 + \epsilon_L(\tfrac{3}{2}g - \tfrac{7}{8})\right.\right.\\ &\qquad \left.\left. - \tfrac{13}{4}\epsilon_R]\,g\,d\eta + 9\epsilon_R f\int_{-\infty}^{\eta} g\,d\eta\right]\right\}\\ &= \frac{1}{h_0}\left(\frac{3\epsilon_R}{4}\right)^{\frac{1}{2}}(1 - \tfrac{5}{8}\epsilon_R + \tfrac{71}{128}\epsilon_R^2)\{x - C_R t + \Theta\},\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{\eta}{2} &= \frac{1}{h_0}\left(\frac{3\epsilon_L}{4}\right)^{\frac{1}{2}}(1 - \tfrac{5}{8}\epsilon_L + \tfrac{71}{128}\epsilon_L^2)\left\{x + C_L t + \frac{h_0}{4}\left(\frac{\epsilon_R}{3}\right)^{\frac{1}{2}}\left[\int_{-\infty}^{\xi}[1 + \epsilon_R(\tfrac{3}{2}f - \tfrac{7}{8})\right.\right.\\ &\qquad \left.\left. - \tfrac{13}{4}\epsilon_L]\,f\,d\xi + 9\epsilon_L g\int_{+\infty}^{\xi} f\,d\epsilon\right]\right\}\\ &= \frac{1}{h_0}\left(\frac{3\epsilon_L}{4}\right)^{\frac{1}{2}}(1 - \tfrac{5}{8}\epsilon_L + \tfrac{71}{128}\epsilon_L^2)\{x + C_L t + \Psi\}.\end{aligned} \tag{54}$$

The term list of (53) and (54) serves to define Θ and Ψ.

The following quantities are presented for convenience of comparison with the experimental results of Chan & Street, and Maxworthy.

(i) *Phase changes:*

$$\left.\begin{aligned}\eta \to -\infty, &\quad \Theta \to 0\\ \xi \to +\infty, &\quad \Psi \to 0\end{aligned}\right\}\ \text{before collision;}$$

$$\left.\begin{aligned}\eta \to +\infty, \quad \Theta &= h_0\left(\frac{\epsilon_L}{3}\right)^{\frac{1}{2}}\left[1+\frac{\epsilon_L}{8}-\frac{13}{4}\epsilon_R\right]+9h_0\left(\frac{\epsilon_L}{3}\right)^{\frac{1}{2}}\epsilon_R f\\ \xi \to -\infty, \quad \Psi &= -h_0\left(\frac{\epsilon_R}{3}\right)^{\frac{1}{2}}\left(1+\frac{\epsilon_R}{8}-\frac{13}{4}\epsilon_L\right]-9h_0\left(\frac{\epsilon_R}{3}\right)^{\frac{1}{2}}\epsilon_L g\end{aligned}\right\} \text{after collision;}$$

$$\Delta\Theta = h_0\left(\frac{\epsilon_L}{3}\right)^{\frac{1}{2}}\left(1+\frac{\epsilon_L}{8}-\tfrac{13}{4}\epsilon_R\right)+9h_0\left(\frac{\epsilon_L}{3}\right)^{\frac{1}{2}}\epsilon_R f, \tag{55}$$

$$\Delta\Psi = -h_0\left(\frac{\epsilon_R}{3}\right)^{\frac{1}{2}}\left(1+\frac{\epsilon_R}{8}-\tfrac{13}{4}\epsilon_L\right)-9h_0\left(\frac{\epsilon_R}{3}\right)^{\frac{1}{2}}\epsilon_L g. \tag{56}$$

The first term in the right-hand side of (55) or the first term in the right-hand side of (56) represents a simple phase shift (a uniform translation without change of wave profile). This agrees, in the first order, with that of Oikawa & Yajima.

(ii) *Distortion of wave profile.* The second term in the right-hand side of (55) and (56) is a function of ξ or η respectively, i.e. different phase shift at different points in the wave. This causes a distortion in the wave profile. For waves of large separation, the localized interaction terms (which are products of f and g) in (50) vanish. Following the right-going wave we then have, after setting $g = 0$,

$$\zeta = \epsilon_R[f+\tfrac{3}{4}\epsilon_R(f^2-f)+\epsilon_R^2(\tfrac{101}{80}f^3-\tfrac{151}{80}f^2+\tfrac{5}{8}f)], \tag{57}$$

with
$$f = \operatorname{sech}^2\left[h_0\left(\frac{3\epsilon_R}{4}\right)^{\frac{1}{2}}(1-\tfrac{5}{8}\epsilon_R+\tfrac{71}{128}\epsilon_R^2)(x-C_R t+\Theta),\right. \tag{58}$$

where the values of Θ before and after collision are given in (i). In figure 2 we plot the perturbed free surface elevation before and after collision according to (57) and (58). Before a collision, $\Theta = 0$, the profile is symmetric but thinner than the first-order approximation given by (58). After the collision, the wave becomes unsymmetrical and tilts backward with respect to the direction of its propagation. We have also plotted the difference between the unsymmetric and the symmetric profile accurate to $O(\epsilon^3)$ (curve 3). The propagation of these tilted waves will be discussed in the next section.

In addition to the tilting discussed above, a second hump will appear in the right branch of the wave profile for values of $\epsilon_R > (48)^{-\frac{1}{4}} \approx 0{\cdot}3799$. However for such an ϵ_R the higher-order terms which are neglected in Θ become comparable to those terms retained. The significance of this hump is thus not clear and we shall say no more about it.

(iii) *Maximum (run-up) amplitude during collision.* For two head-on colliding solitary waves with their maximum heights before collision defined as ϵ_R and ϵ_L, the run-up at a point (ξ, η) is defined by the value of the perturbed free surface elevation of (50). It is easy to see that the maximum run-up exists at the point (ξ, η) where $f = 1$, $g = 1$ and hence

$$\text{maximum run-up} = \epsilon_R+\epsilon_L+\frac{\epsilon_R\epsilon_L}{2}+\tfrac{3}{8}\epsilon_R\epsilon_L(\epsilon_R+\epsilon_L). \tag{59a}$$

For two identical solitary waves $\epsilon_R = \epsilon_L$ and

$$\text{maximum run-up} = 2\epsilon_R+\frac{\epsilon_R^2}{2}+\tfrac{3}{4}\epsilon_R^3. \tag{59b}$$

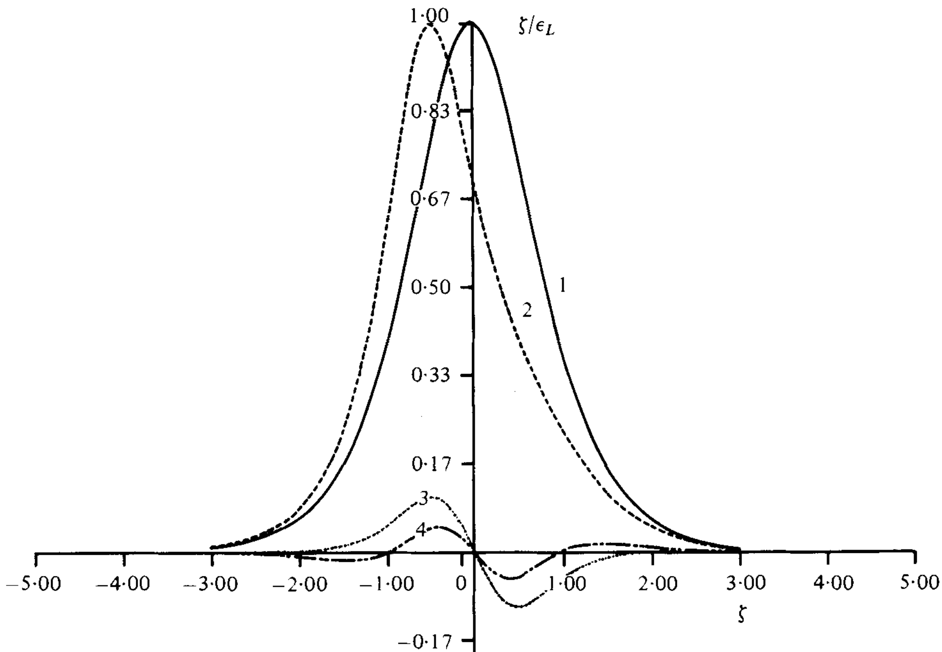

FIGURE 2. Wave profile ζ/ϵ_L *vs.* ξ for $\epsilon_L = 0{\cdot}2$. 1, ——, before collision; 2, – – –, after collision; 3, · · · · · ·, $v_0 = 9\epsilon_L^2 SS'$; 4, — · · —, wavelet at birth.

In figure 1 we plot the maximum run-up of (59*b*), for the first-, second- (Byatt-Smith), and third-order approximations, respectively. The existing numerical (Chan & Street 1970), as well as the experimental, results (Chan & Street 1970; Maxworthy 1976) are also presented (in the plot). It is seen that our third-order result is in complete agreement with the numerical values of Chan & Street for $\epsilon_R \leqslant 0{\cdot}5$. The experimental data has a sizable spread. Even then, the third-order result seems to represent them best if one excludes those data of Maxworthy which represent wave–wave interaction experiments.

5. Time evolution of the unsymmetrical waves

It is seen from the results of the previous section that the after-effect of a collision on a solitary wave is manifested only by the phase functions θ and ϕ. As the collision process comes to an end designated by $\eta \to +\infty$ and $\xi \to -\infty$ for the right- and left-going wave, respectively, we see that, for the right-going wave say, all the terms associated with $g(\eta)$ vanish, except those appearing in the phase function where g is under an integral sign. After dropping all terms which are products of f and g, we are left with a solitary wave propagating with a constant speed. This wave, however, is not a solution to the equation for a wave propagating without change of speed and shape. In this section we consider the slow time evolution of this unsymmetrical wave. We first derive the appropriate governing equation, and then solve this with the wave profile emerging from the collision as initial data.

Since the ξ dependence of θ first occurs in the third order of ϵ, we take that a slow time dependence does not come in until that order of approximation. We assume, following the right-going wave, that

$$\zeta = \epsilon S(\xi_0) + \epsilon^2 \zeta_2(\xi_0) + \epsilon^3 \zeta_3(\xi_0, \tau) + \ldots, \tag{60}$$

$$w = \epsilon C_R[S(\xi_0) + \epsilon W_2'(\xi_0) + \epsilon^2 W_3'(\xi_0, \tau) + \ldots], \tag{61}$$

where $\xi_0 = \epsilon^{\frac{1}{2}} k(x - C_R t)$, τ in ζ_3 and W_3' allows the slow time evolution of the wave in a moving frame fixed in the right-going wave. We have also assumed that the left-going wave is far away and exerts no more effect on the right-going wave. We define the slow time variable by

$$\tau = \epsilon^{\frac{3}{2}} k C_R t. \tag{62}$$

Substituting (62), (61), and (60) into the basic equations (6) and (7), we obtain equations for S in the first-order approximation, and ζ_2, W_2' for the order in ϵ^2. The solutions of these equations are

$$\left.\begin{aligned} R_1 &= \tfrac{1}{2}, \quad S = \operatorname{sech}^2 \tfrac{1}{2}\xi_0, \\ R_2 &= \tfrac{19}{40}, \quad W_2' = -S^2 + S, \quad \zeta_2 = \tfrac{3}{4}S^2 + \tfrac{1}{2}S. \end{aligned}\right\} \tag{63}$$

These are equivalent to the solutions obtained by Laitone.

To the third order of approximation, we have, in terms of W_3',

$$\frac{\partial^2 W_3'}{\partial \xi_0^2} + (3S-1)\, W_3' + 2\int^{\xi_0} \frac{\partial W_3'}{\partial \tau}\, d\xi_0 = (2R_3 - \tfrac{55}{56})\, S - \tfrac{27}{5} S^2 + 14S^3 - 9S^4. \tag{64}$$

We decompose $W_3'(\xi_0, \tau)$ above into a stationary solution plus a transient, i.e.

$$W_3'(\xi_0, \tau) = W_3(\xi_0) + V(\xi_0, \tau). \tag{65}$$

These functions then satisfy

$$\frac{d^2 W_3}{\partial \xi_0^2} + (3S-1)\, W_3 = (2R_3 - \tfrac{55}{56})\, S - \tfrac{27}{5} S^2 + 14S^3 - 9S^4, \tag{66}$$

and

$$2\frac{\partial V}{\partial \tau} + \frac{\partial}{\partial \xi_0}\left[\frac{\partial^2 V}{\partial \xi_0^2} + (3S-1)\, V\right] = 0. \tag{67}$$

The solutions to (66) are

$$W_3 = \tfrac{6}{5}S^3 - \tfrac{11}{5}S^2 + \tfrac{4}{5}S, \quad R_3 = \tfrac{55}{112}, \tag{68}$$

which is equivalent to the third-order solution of Grimshaw (1971).

We now consider the time evolution of (67) with the initial data of the difference between the wave field of the right-going wave immediately after collision and that of $W_3(\xi_0)$ of (68), i.e. curve 3 in figure 2. It can be shown that, for any μ and k,

$$V_\mu(\xi_0, \tau) = F(\xi_0, k)\, e^{-\frac{1}{2}i\mu\tau + ik\xi_0}, \tag{69}$$

is a solution of (67) provided that we take

$$F(\xi_0, k) = ik(k^2-1) + 2ikS(\xi_0) + S'(\xi_0) + 2k^2 \frac{S'(\xi_0)}{S(\xi_0)}, \tag{70}$$

and $k^3 + k + \mu = 0$.

This form of solution is first suggested by Jeffrey & Kakutani (1970). Since $F(\xi_0, k)$ is bounded for all ξ_0, to have a bounded solution $V_\mu(\xi_0, \tau)$ in ξ_0, we must take a real k, which in turn requires a real μ. Therefore the general solution for V in (67) can be represented as an integral over all real k, i.e.

$$V(\xi_0, \tau) = \int_{-\infty}^{\infty} dk\, A(k)\, F(\xi_0, k)\, e^{-\frac{1}{2}i\mu\tau + ik\xi_0}, \tag{71}$$

where $A(k)$ is determined by the initial data $V(\xi_0, \tau = 0)$, i.e.

$$V(\xi_0, 0) = \int_{-\infty}^{\infty} dk\, A(k)\, F(\xi_0, k)\, e^{ik\xi_0}.$$

Substituting $F(\xi_0, k)$ of (70) into the above equation, we find that

$$V(\xi_0, 0) = \left[-\frac{d^3}{d\xi_0^3} - 2\frac{S'}{S}\frac{d^2}{d\xi_0^2} + (2S-1)\frac{d}{d\xi_0} + S'\right] \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\xi_0}. \tag{72}$$

The general solution of (72) can be expressed as repeated integrals, i.e.

$$\int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\xi_0} = -\frac{1}{S(\xi_0)} \int^{\xi_0} S(\xi_1)\, d\xi_1 \int^{\xi_1} \frac{d\xi_2}{S(\xi_2)} \int^{\xi_2} S(\xi_3)\, V_0(\xi_3, 0)\, d\xi_3.$$

However, for $V(\xi_0, 0) = \gamma S(\xi_0)\, S'(\xi_0)$, which is what we have as initial data up to $O(\epsilon^3)$ for the collision of two solitary waves, it is easily verified that

$$\int_{-\infty}^{\infty} dk\, A_p(k)\, e^{ik\xi_0} = \frac{\gamma}{9}[4 + S(\xi_0)]. \tag{73}$$

The value of γ is determined by (53) and (55) to be $9\epsilon_R^{\frac{3}{2}}\epsilon_L^{\frac{1}{2}}$. By Fourier integral theorem, we obtain

$$A_p(k) = \frac{\gamma}{9}[4\delta(k) + \tilde{S}(k)], \tag{74}$$

where $\delta(k)$ is the Dirac δ-function and

$$\tilde{S}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-ik\xi}\, S(\xi) = \frac{2k}{\sinh \pi k}. \tag{75}$$

Since $S(\xi)$ vanishes exponentially as $\xi \to \pm\infty$ we see that $\tilde{S}(k)$ is an analytic function of k inside a strip containing the real axis. Substituting the function $A_p(k)$ back into (71), we then obtain the solution of (67) corresponding to an initial data $\gamma SS'$ as

$$V(\xi_0, \tau) = \tfrac{4}{9}\gamma S'(\xi_0) + \frac{\gamma}{9}\int_{-\infty}^{\infty} dk\, \tilde{S}(k)\, F(\xi_0, k)\, e^{\frac{1}{2}i\tau(k+k^3) + ik\xi_0}. \tag{76}$$

The asymptotic behaviour of the integral in (76) is given by

$$\left(\frac{8\pi}{3|k_0|\tau}\right)^{\frac{1}{2}} \tilde{S}(k_0)\left\{\left[1 + \frac{2k_0^2}{S(\xi_0)}\right] S'(\xi_0) \operatorname{Cos}\left(\tau k_0^3 - \frac{\pi}{4}\right) + k_0[k_0^2 - 1 + 2S|\xi_0|] \sin\left(\tau k_0^3 - \frac{\pi}{4}\right)\right. \tag{77}$$

where $3k_0^2 = -(2\xi_0/\tau + 1)$. As $\tau \to \infty$, this represents a group of waves trailing behind its mother solitary wave. The amplitude of the wave group diminishes in time owing to dispersion. This wavelet, starting out with a profile of $\gamma SS' - \frac{4}{9}\gamma S'$ plotted as the curve 4 in figure 2, tends to zero asymptotically as $\tau \to \infty$ owing to dispersion.

The first term in (76) is independent of τ. We note that S' is a complementary solution of (66). This solution represents a uniform phase shift, as we have mentioned in the paragraph immediately after (32).

To calculate the total (uniform) phase shift of the right-going wave up to $O(\epsilon^{\frac{3}{2}})$, we replace the last term in (55) by $4\epsilon_R$ which is the phase shift due to the first term in (76), i.e.

$$\Delta\theta = h_0\left(\frac{\epsilon_L}{3}\right)^{\frac{1}{2}}(1+\tfrac{1}{8}\epsilon_L+\tfrac{3}{4}\epsilon_R). \tag{78}$$

A similar formula for the left-going wave is

$$\Delta\phi = -h_0\left(\frac{\epsilon_R}{3}\right)^{\frac{1}{2}}(1+\tfrac{1}{8}\epsilon_R+\tfrac{3}{4}\epsilon_L). \tag{79}$$

In conclusion, we state three main results from our calculations in relation to the existing experiments:

(1) The maximum amplitude during the collision is given by the formula (59*a*). This is plotted in figure 1 together with the experimental results of Chan & Street and Maxworthy. The third-order result seems to agree perfectly with the numerical calculation of Chan & Street. They also agree very well with the reflexion experiments of both Chan & Street (1970) and Maxworthy (1976). They are consistently lower than the wave–wave experiment of Maxworthy.

(2) The total phase shifts are given by (78) and (79). These represent a retardation of the waves during their collision. The $\epsilon^{\frac{3}{2}}$ correction terms make the calculated phase shift a little closer to the value obtained by Maxworthy's experiment as shown in figure 3. However, the theory does not seem able to account for the experimental result which measured amplitude-independent phase shifts.

(3) Each solitary wave sheds a secondary wave. For the right-going wave, the profile of the secondary wave at time of shedding is given by the curve 4 in figure 2 (accurate up to the order of ϵ^3). These secondary waves propagate in the opposite direction of their parent waves. Their amplitudes decrease in time owing to dispersion. Maxworthy (1976) indicates appearance of the secondary wave in his reflexion experiment. A quantitative measurement will be of great interest for comparison.

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Appendix

Substituting (18) into (17), we obtain a power series in ϵ. Here we recast each order of ϵ alone:

$$O(\epsilon) = 0: \quad \frac{\partial \alpha_0}{\partial \eta} = 0, \quad \frac{\partial \beta_0}{\partial \xi} = 0. \tag{A 1}$$

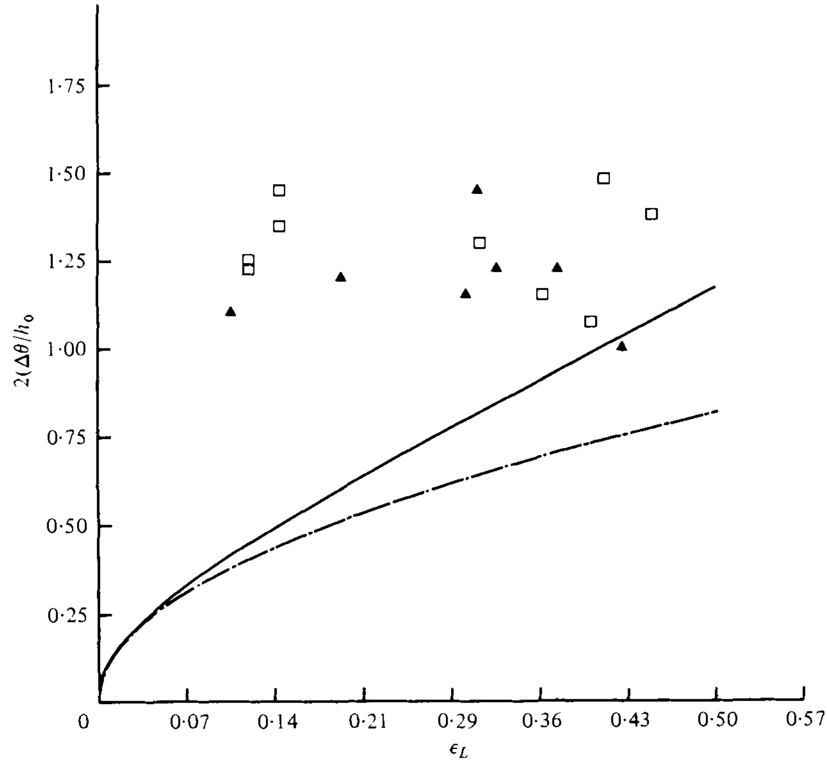

FIGURE 3. Magnitude of twice phase shift ($2\Delta\theta/h_0$) *vs.* wave amplitude $\Delta h/h_0 = \epsilon_L$. ——, second order; – – –, Oikawa & Yajima (1973) and implicit result from Byatt-Smith (1971). △, Maxworthy's (1976) wave–wave interaction; □, end-wall reflexion.

$$O(\epsilon^2) = 0: \quad 4l\frac{\partial\alpha_1}{\partial\eta}+l\frac{\partial}{\partial\eta}\left[\frac{2h_0^2l^2}{3}\frac{\partial^2\beta_0}{\partial\eta^2}-\left(\alpha_0+\frac{\beta_0}{2}\right)\beta_0\right]+4lk\frac{\partial\alpha_0}{\partial\xi}\left[\frac{\partial\theta_0}{\partial\eta}-\frac{1}{4l}\beta_0\right]$$
$$+k\frac{\partial}{\partial\xi}\left[\frac{h_0^2k^2}{3}\frac{\partial^2\alpha_0}{\partial\xi^2}+(\tfrac{3}{2}\alpha_0-2R_1a)\,\alpha_0\right]=0. \qquad \text{(A 2)}$$

The equation for β_1 is obtained from (A 2) by replacing α_1 by β_1, α_0 by β_0, l by k, θ_0 by ϕ_0, η by ξ, a by b, and R_1 by L_1.

$$O(\epsilon^3) = 0: \quad 4l\frac{\partial\alpha_2}{\partial\eta}+l\frac{\partial}{\partial\eta}\left[\frac{2h_0^2l^2}{3}\frac{\partial^2G_1^*}{\partial\eta^2}-(\alpha_0+\beta_0)\,G_1^*\right.$$
$$-\beta_0F_1^*+\left(-\tfrac{3}{2}\beta_0^3-\tfrac{3}{8}b\beta_0^2-\tfrac{65}{8}\beta_0^2\alpha_0+\tfrac{7}{8}\beta_0\alpha_0^2\right.$$
$$\left.\left.+\tfrac{11}{2}b\alpha_0\beta_0-\frac{a\alpha_0\beta_0}{2}+\tfrac{4}{5}b^2\beta_0\right)\right]$$
$$+4kl\frac{\partial\alpha_0}{\partial\xi}\left[\frac{\partial\theta_1}{\partial\eta}-\frac{1}{4l}G_1^*-\frac{1}{4l}(9\alpha_0\beta_0+\tfrac{13}{8}\beta_0^2-\tfrac{13}{4}a\beta_0-\tfrac{5}{4}b\beta_0)\right]$$
$$+k\frac{\partial}{\partial\xi}\left[\frac{h_0^2k^2}{3}\frac{\partial^2F_1^*}{\partial\xi^2}+(3\alpha_0-a)\,F_1^*-(2R_2-\tfrac{19}{20})\,a^2\alpha_0\right.$$
$$\left.-\tfrac{9}{8}a\alpha_0^2-\frac{\alpha_0^3}{4}\right]=0. \qquad \text{(A 3)}$$

The equation for β_2 can be obtained from (A 3) in a similar fashion as β_1 from (A 2).

$$O(\epsilon^4) = 0: \quad 4l\frac{\partial \alpha_3}{\partial \eta} + l\frac{\partial}{\partial \eta}[\dots] + 4kl\frac{\partial \alpha_0}{\partial \xi}[\dots]$$

$$+k\frac{\partial}{\partial \xi}\left[\frac{h_0^2 k^2}{3}\frac{\partial^2 F_2^*}{\partial \xi^2} + (3\alpha_0 - a)F_2^* - (2R_3 - \tfrac{55}{56})a^3\alpha_0\right.$$

$$\left. + \tfrac{393}{160}a^2\alpha_0^2 - \tfrac{201}{16}a\alpha_0^3 + \tfrac{591}{64}\alpha_0^4\right] = 0, \quad \text{(A 4)}$$

where $F_1^* = a^2F_1$, $G_1^* = b^2G_1$, $G_2^* = b^3G_2$ and $F_2^* = a^3F_2$; F_1 and F_2 are arbitrary functions of integration of (A 2) and (A 3) for α_1 and α_2; G_1 and G_2 are the corresponding arbitrary functions of integration for β_1 and β_2. The symbols [...] in (A 4) represent all the non-secular terms in the (ϵ^4) order.

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